AP Calculus, Semester 1
Semester Final
Instructor's Version B

1. The radius, r, of a sphere is increasing at a constant rate of 0.05 meters per second.

A. At the time when the radius of the sphere is 12 meters, what is the rate of increase in its volume? (12 points)

Answer: $V = \frac{4}{3}\pi r^3$ (1 point) $\Rightarrow$ $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$ (4 points) $\Rightarrow$ $\frac{dV}{dt} = 4\pi(12)^2(0.05)$

(4 points) $\Rightarrow$ $\frac{dV}{dt} = \frac{144}{5}\pi$ m^3/sec (3 points)

B. At the time when the volume of the sphere is 36π cubic meters, what is the rate of increase in its surface area? (15 points)

Answer: $V = \frac{4}{3}\pi r^3 = 36\pi$ (3 points) $\Rightarrow$ $r^3 = 27$ (2 points) $\Rightarrow$ $r = 3$ (1 point)

$S = 4\pi r^2$ (1 point) $\Rightarrow$ $\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$ (3 points) $\Rightarrow$ $\frac{dS}{dt} = 8\pi(3)(0.05) = \frac{6\pi}{5}$ m^2/sec (5 points)

C. Express the rate at which the volume of the sphere changes with respect to the surface area of the sphere (as a function of r). (6 points)

Answer: $\frac{dV}{dS} = \frac{dV/dr}{dS/dr} = \frac{4\pi r^2}{8\pi r} = \frac{r}{2}$ cubic meters per square meter

(6 points: 3 points for recognizing the rate that is being asked for, 3 points for the answer. The units are not required for full credit here.)

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2. A particle moves along the *x*-axis so that its velocity at any time $t \geq 0$ is given by $v(t)=(2\pi-5)t-\sin(\pi t)$.

A. Find the acceleration at any time *t*. (6 points)

Answer: $a(t)=(2\pi-5)-\pi\cos(\pi t)$ (1 point for taking the derivative of the first term correctly, 2 points for taking the derivative of sine correctly, 3 points for using the Chain Rule.)

B. Find the minimum acceleration of the particle over the interval [0, 3]. (12 points)

Answer: $a'(t)=\pi^2\sin(\pi t)=0$ for t = 0, 1, 2, and 3. (6 points: 3 points for using $a'(t)$ = 0, 3 points for solving) The sign of $a'(t)$ changes from negative to positive at 0 and 2, so those are the locations of minimum values (3 points). The minimum acceleration is π - 5 (3 points).

C. Find the maximum velocity of the particle over the interval [0, 2]. (12 points)

Answer: $a(t)=(2\pi-5)-\pi\cos(\pi t)$ = 0 at t = 0.366, and 1.634. (6 points: 3 points for using $a(t)$ = 0, 3 points for solving) The sign of $a(t)$ changes from positive to negative at 1.634, so that is the location of the maximum value (3 points). The maximum velocity is 3.009 (3 points).

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3. Consider the function $h(x)=a(-2x+1)^5-b$, where $a\neq 0$ and $b\neq 0$ are constants.

A. Find $h'(x)$ and $h''(x)$. (6 points)

Answer: $h'(x)=5a(-2x+1)^4(-2)=-10a(-2x+1)^4$ (3 points) and $h''(x)=80a(-2x+1)^3$ (3 points)

B. Show that *h* is monotonic (that is, that either *h* always increases or remains constant or *h* always decreases or remains constant). (12 points)

Answer: The factor $(-2x + 1)^4$ in the first derivative is always positive, or 0 (4 points). If *a* is negative, $-10a$ will be positive meaning that *h* will always increase or be constant. (4 points) If *a* is positive, $-10a$ will be negative, meaning *h* will always decrease or remain constant. (4 points).

C. Show that the *x*-coordinate(s) of the location(s) of the critical points are independent of a and b. (12 points)

Answer: The critical values occur whenever the first derivative equals zero or is undefined. (3 points) Because the first derivative does not contain b, the location(s) of the critical number(s) are not dependent on b. (3 points) The first derivative is never undefined (3 points). The first derivative only equals zero whenever the factor $(-2x + 1)^4$ equals zero, since a itself cannot equal zero. (3 points). Therefore, the location(s) of the critical numbers do not depend on a.

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4. Consider $g(x)=\begin{cases} a\sin x+b, & if \ \ x\le 2\pi \\ x^2-\pi x+2, & if \ \ x>2\pi \end{cases}$.

A. Find the values of a and b such that $g(x)$ is a differentiable function. (12 points)

Answer: First, $g(x)$ must be continuous, so: $a\sin(2\pi)+b=(2\pi)^2-2\pi^2+2$ (3 points) and $b=2\pi^2+2$ (3 points). Then, the one-sided derivatives must equal, so: $a\cos(2\pi)=2(2\pi)-\pi$ (3 points), and $a=3\pi$ (3 points).

B. Write the equation of the tangent line to $g(x)$ at $x = 2\pi$. (9 points)

Answer: First, $g(2\pi)=2\pi^2+2$. (3 points) Also, $g'(2\pi)=3\pi$. (3 points)
So the tangent line equation is: $y - (2\pi^2 + 2) = 3\pi(x - 2\pi)$. (3 points)

C. Use the tangent line equation from part B to write an approximation for the value of $g(6)$. Do not simplify. (6 points)

Answer: The tangent line equation is: $y - (2\pi^2 + 2) = 3\pi(x - 2\pi)$,
so $y - (2\pi^2 + 2) = 3\pi(6 - 2\pi)$ (3 points) and $y = (2\pi^2 + 2) + 3\pi(6 - 2\pi)$ (3 points).

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